Philadelphia University
Faculty of Science
Department of Basic Sciences and Mathematics

Real Analysis 2 First Exam

Student name: ___________________________ Number:_________________

Remember that: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

1) If $f(x) = \sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, show that $(f^{-1})'(y) = \frac{1}{\sqrt{1-y^2}}$ for $y \in [-1,1]$.

2)
   a) State Mean Value theorem
   b) Use Mean Value theorem to prove that $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$.

3) Suppose that $f: [0,2] \to \mathbb{R}$ is continuous on $[0,2]$ and differentiable on$(0,2)$, and that $f(0) = 0, f(1) = 1, f(2) = 1$.

(a) Show that there exists $c_1 \in (0,1)$ such that $f'(c_1) = 1$
(b) Show that there exists $c_2 \in (0,2)$ such that $f'(c_2) = \frac{1}{3}$

4) Let $f: I \to \mathbb{R}$ be differentiable on the interval I. Show that if f' is positive on I, then f is strictly increasing on I.

5) Evaluate

$$\lim_{x \to 0} \frac{x^2 - (\sin x)^2}{x^4}$$

6) Let $f: I \to \mathbb{R}$ be bounded. If P_1, P_2 are any two partitions of I, prove that $L(f, P_1) \le U(f, P_2)$.

7) Let $f(x) = x^2 \; for \; x \in [0,2]$. For the partition $P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$
   a) Calculate $L(f, P_n) \; and \; U(f, P_n)$.
   b) Is f Riemann integrable? If yes find $\int_0^1 f$